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COMPUTER ARITHMETIC, CHAOS AND FRACTALS

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In this paper we explore aspects of computer arithmetic from the viewpoint of dynamical systems. We demonstrate the effects of finite precision arithmetic in three uniformly hyperbolic chaotic dynamical systems: Bernoulli shifts, cat maps, and pseudorandom number generators. We show that elementary floating-point operations in binary computer arithmetic possess an inherently fractal structure. Each of these dynamical systems allows us to compare the exact results in integer arithmetic with those obtained by using floating-point arithmetic.

1. Introduction

In this paper we view the computer as a dynamical system. We investigate the effects of computer arithmetic on computations in deterministic chaotic dynamical systems.

We show that the computation of orbits in highly unstable systems – deterministic chaotic dynamical systems – can be affected greatly by the use of the floating-point machine representations to mimic the real number field. We have found that the use of finite precision arithmetic in binary floating-point representations introduces another sensitivity into calculations in these highly unstable systems. Finite machine representations preclude the exact specifications of numbers that are not dyadic rationals. For this reason, even the act of entering data into a program may induce errors from the very beginning of the calculation. Differences in the least significant bit between the data entered and the machine representation begin to propagate immediately into the mainstream calculation of the chaotic orbit. This error propagation destroys validity of conclusions based on an inferred relation between the data entered and the output achieved.

A fundamental difference between the usual form of calculation in a chaotic dynamical system and what we describe here is that we couple the orbit calculation with conditional branching whenever a prescribed condition is met. The use of conditional branching in orbit computations is usually not considered in dynamical systems theory for those systems having uniqueness.

We can generate a pseudo-orbit using branching by the following mappings and conditions:

$$x \to 3x \mod 1$$

when $n \otimes 1.0/n = 1.0$ and

$$x \to 3x + \varepsilon \mod 1$$

when $n \otimes 1.0/n \neq 1.0$.

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